

# On the evolution of density perturbations in $f(R)$ theories of gravity

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In the context of  $f(R)$  theories of gravity, we study the evolution of scalar cosmological perturbations in the metric formalism. Using a completely general procedure, we find the exact fourth-order differential equation for the matter density perturbations in the longitudinal gauge. In the case of sub-Hubble modes, the expression reduces to a second-order equation which is compared with the standard (quasi-static) equation used in the literature. We show that for general  $f(R)$  functions the quasi-static approximation is not justified. However, for those functions adequately describing the present phase of accelerated expansion and satisfying local gravity tests, it provides a correct description for the evolution of perturbations.

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## INTRODUCTION

The present phase of accelerated expansion of the universe [1] poses one of the most important problems of modern cosmology. It is well known that ordinary Einstein's equations in either a matter or radiation dominated universe give rise to decelerated periods of expansion. In order to have acceleration, the total energy-momentum tensor appearing on the right hand side of the equations should be dominated at late times by a hypothetical negative pressure fluid usually called dark energy (see [2] and references therein).

However, there are other possibilities to generate a period of acceleration in which no new sources are included on the r.h.s. of the equations, but instead Einstein's gravity itself is modified [3]. In one of such possibilities, new functions of the curvature scalar ( $f(R)$  terms) are included in the gravitational action, which amounts to modifying the l.h.s of the equations of motion. Although such theories are able to describe the accelerated expansion on cosmological scales correctly, they typically give rise to strong effects on smaller scales. In any case viable models can be constructed to be compatible with local gravity tests and other cosmological constraints [4].

The important question that arises is therefore how to discriminate dark energy models from modified gravities using present or future observations. It is known that by choosing particular  $f(R)$  functions, one can mimic any background evolution (expansion history), and in particular that of  $\Lambda$ CDM. Accordingly, the exclusive use of observations such as high-redshift Hubble diagrams from SNIa [1], baryon acoustic oscillations [5] or CMB shift factor [6], based on different distance measurements which are sensitive only to the expansion history, cannot settle the question of the nature of dark energy [7].

However, there exists a different type of observations which are sensitive, not only to the expansion history, but also to the evolution of matter density perturbations. The fact that the evolution of perturbations depends on the specific gravity model, i.e. it differs in general from that of Einstein's gravity even though the background evolution is the same, means that this kind of observations will help distinguishing between different models for acceleration.

In this work we study the problem of determining the exact equation for the evolution of matter density perturbations for arbitrary  $f(R)$  theories. Such problem had been previously considered in the literature ([8, 9, 10, 11, 12, 13]) and approximated equations have been widely used. They are typically based on the so called quasi-static approximation in which all the time derivative terms for the gravitational potentials are discarded, and only those including density perturbations are kept [14]. From our exact result, we will be able to determine under which conditions such an approximation can be justified.

The paper is organized as follows: in Section 2, we briefly review the perturbations equations for the standard  $\Lambda$ CDM model. In Section 3 we obtain the perturbed equations for general  $f(R)$  theories. In Section 4 we describe the procedure to obtain the general equation for the density perturbation. In Section 5 we summarize the main viability condition for  $f(R)$  theories. Section 6 is devoted to the study of the validity of the quasi-static approximation. In Section 7 we apply our results to some particular models and finally in Section 8 we include the main conclusions. In Appendices I and II we have also included complete expressions for the relevant coefficients of the perturbation equation.

## DENSITY PERTURBATIONS IN $\Lambda$ CDM

Let us start by considering the simplest model for dark energy described by a cosmological constant  $\Lambda$ . The cor-

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responding Einstein's equations read:

$$G^\mu{}_\nu = -8\pi G T^\mu{}_\nu - \Lambda \delta^\mu{}_\nu \quad (1)$$

where  $G^\mu{}_\nu$  is the Einstein's tensor and  $T^\mu{}_\nu$  is the energy-momentum tensor for matter.

In the metric formalism for the  $\Lambda$ CDM model it is possible to obtain a second order differential equation for the growth of matter density perturbation  $\delta \equiv \delta\rho/\rho_0$ . Let us consider the scalar perturbations of a flat FRW metric in the longitudinal gauge and in conformal time:

$$ds^2 = a^2(\eta)[(1 + 2\Phi)d\eta^2 - (1 - 2\Psi)(dr^2 + r^2 d\Omega_2^2)] \quad (2)$$

where  $\Phi \equiv \Phi(\eta, \vec{x})$  and  $\Psi \equiv \Psi(\eta, \vec{x})$  are the scalar perturbations. From this metric, we obtain the first-order perturbed Einstein's equation:

$$\delta G^\mu{}_\nu = -8\pi G \delta T^\mu{}_\nu \quad (3)$$

where the perturbed energy-momentum tensor reads:

$$\begin{aligned} \delta T^0_0 &= \delta\rho = \rho_0\delta, \quad \delta T^i_j = -\delta P \delta^i_j = -c_s^2 \delta^i_j \rho_0\delta \\ \delta T^0_i &= -\delta T^i_0 = -(1 + c_s^2)\rho_0\partial_i v \end{aligned} \quad (4)$$

with  $\rho_0$  the unperturbed energy density and  $v$  the potential for velocity perturbations. We assume that the perturbed and unperturbed matter have the same equation of state, i.e.  $\delta P/\delta\rho \equiv c_s^2 \equiv P_0/\rho_0$ , where  $c_s = 0$  for matter perturbations. The resulting differential equation for  $\delta$  in Fourier space is written as:

$$\begin{aligned} \delta'' + \mathcal{H} \frac{k^4 - 6\tilde{\rho}k^2 - 18\tilde{\rho}^2}{k^4 - \tilde{\rho}(3k^2 + 9\mathcal{H}^2)} \delta' - \\ - \tilde{\rho} \frac{k^4 + 9\tilde{\rho}(2\tilde{\rho} - 3\mathcal{H}^2) - k^2(9\tilde{\rho} - 3\mathcal{H}^2)}{k^4 - \tilde{\rho}(3k^2 + 9\mathcal{H}^2)} \delta = 0 \end{aligned} \quad (5)$$

where  $\tilde{\rho} \equiv 4\pi G \rho_0 a^2 = -\mathcal{H}' + \mathcal{H}^2$  and  $\mathcal{H} \equiv a'/a$  with prime denoting derivative with respect to time  $\eta$ . We point out that it is not necessary to explicitly calculate potentials  $\Phi$  and  $\Psi$  to obtain equation (5), but algebraic manipulations in the field equations are enough to get this result. In the extreme sub-Hubble limit, i.e.  $k\eta \gg 1$  or equivalently  $k \gg \mathcal{H}$ , (5) is reduced to the well-known expression:

$$\delta'' + \mathcal{H}\delta' - 4\pi G \rho_0 a^2 \delta = 0 \quad (6)$$

In this regime and at early times, the matter energy density dominates over the cosmological constant and it is easy to show that  $\delta$  solutions for (6) grow as  $a(\eta)$ . At late times (near today) the cosmological constant contribution is not negligible and power-law solutions for (6) no longer exist. It is necessary in this case to assume an ansatz for  $\delta$ . One which works very well is the one proposed in [7] and [15]

$$\frac{\delta(a)}{a} = e^{\int_{a_i}^a [\Omega_m(a)^\gamma - 1] d \ln a} \quad (7)$$

This expression fits with high precision the numerical solution for  $\delta$  with a constant parameter  $\gamma = 6/11$ .

## PERTURBATIONS IN $f(R)$ THEORIES

Let us consider the modified gravitational action:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R + f(R)) \quad (8)$$

where  $R$  is the scalar curvature<sup>1</sup>. The corresponding equations of motion read:

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{2}g_{\mu\nu}f(R) + R_{\mu\nu}f_R(R) - g_{\mu\nu}\square f_R(R) + f_R(R)_{;\mu\nu} \\ &= -8\pi G T_{\mu\nu} \end{aligned} \quad (9)$$

where  $f_R(R) = df(R)/dR$ . For the background flat Robertson-Walker metric they read:

$$\frac{3\mathcal{H}'}{a^2}(1 + f_R) - \frac{1}{2}(R_0 + f_0) - \frac{3\mathcal{H}}{a^2}f'_R = -8\pi G \rho_0 \quad (10)$$

and

$$\begin{aligned} \frac{1}{a^2}(\mathcal{H}' + 2\mathcal{H}^2)(1 + f_R) - \frac{1}{2}(R_0 + f_0) - \frac{1}{a^2}(\mathcal{H}f'_R + f''_R) \\ = 8\pi G c_s^2 \rho_0 \end{aligned} \quad (11)$$

where  $R_0$  denotes the scalar curvature corresponding to the unperturbed metric,  $f_0 \equiv f(R_0)$ ,  $f_R \equiv df(R_0)/dR_0$  and prime means derivative with respect to time  $\eta$ . A very useful equation to use in the following calculations is the (11) – (10) combination

$$2(1 + f_R)(-\mathcal{H}' + \mathcal{H}^2) + 2\mathcal{H}f'_R - f''_R = 8\pi G \rho_0(1 + c_s^2)a^2 \quad (12)$$

Finally we have the conservation equation:

$$\rho'_0 + 3(1 + c_s^2)\mathcal{H}\rho_0 = 0 \quad (13)$$

Using the perturbed metric (2) and the perturbed energy-momentum tensor (4), the first order perturbed equations, assuming that the background equations hold, may be written as:

$$\begin{aligned} (1 + f_R)\delta G^\mu{}_\nu + (R_0^\mu{}_\nu + \nabla^\mu \nabla_\nu - \delta^\mu_\nu \square)f_{RR}\delta R + \\ [(\delta g^{\mu\alpha})\nabla_\nu \nabla_\alpha - \delta^\mu_\nu (\delta g^{\alpha\beta})\nabla_\alpha \nabla_\beta]f_R - \\ [g_0^{\alpha\mu}(\delta\Gamma^\gamma_{\alpha\nu}) - \delta^\mu_\nu g_0^{\alpha\beta}(\delta\Gamma^\gamma_{\beta\alpha})]\partial_\gamma f_R = -8\pi G \delta T^\mu{}_\nu \end{aligned} \quad (14)$$

where  $f_{RR} = d^2f(R_0)/dR_0^2$ ,  $\square \equiv \nabla_\alpha \nabla^\alpha$  and  $\nabla$  is the usual covariant derivative with respect to the unperturbed FRW metric (see [16] for perturbed metric, connection symbols and other useful perturbed quantities). Notice that unlike the ordinary Einstein-Hilbert

<sup>1</sup> The Riemann tensor definition is  $R^\mu{}_{\nu\alpha\beta} = \partial_\beta \Gamma^\mu_{\nu\alpha} - \partial_\alpha \Gamma^\mu_{\nu\beta} + \Gamma^\mu_{\sigma\beta} \Gamma^\sigma_{\nu\alpha} - \Gamma^\mu_{\sigma\alpha} \Gamma^\sigma_{\nu\beta}$  which has opposite sign to the one proposed in [16].

case, with second order equations, this is a set of fourth-order differential equations. By computing the covariant derivative with respect to the perturbed metric  $\tilde{\nabla}$  of the perturbed energy-momentum tensor  $\tilde{T}_\nu^\mu$ , we find the conservation equations:

$$\tilde{\nabla}_\mu \tilde{T}_\nu^\mu = 0 \quad (15)$$

which do not depend on  $f(R)$ .

For the linearized Einstein's equations, the components  $(00)$ ,  $(ii)$ ,  $(0i) \equiv (i0)$  and  $(ij)$ , where  $i, j = 1, 2, 3$ ,  $i \neq j$ , in Fourier space, read respectively:

$$(1 + f_R) \begin{bmatrix} -k^2(\Phi + \Psi) - 3\mathcal{H}(\Phi' + \Psi') + (3\mathcal{H}' - 6\mathcal{H}^2)\Phi - 3\mathcal{H}'\Psi \\ + f_R'(-9\mathcal{H}\Phi + 3\mathcal{H}\Psi - 3\Psi') \end{bmatrix} = 2\tilde{\rho}\delta \quad (16)$$

$$(1 + f_R) \begin{bmatrix} \Phi'' + \Psi'' + 3\mathcal{H}(\Phi' + \Psi') + 3\mathcal{H}'\Phi + (\mathcal{H}' + 2\mathcal{H}^2)\Psi \\ + f_R'(3\mathcal{H}\Phi - \mathcal{H}\Psi + 3\Psi') + f_R''(3\Phi - \Psi) \end{bmatrix} = 2c_s^2\tilde{\rho}\delta \quad (17)$$

$$(1 + f_R)[\Phi' + \Psi' + \mathcal{H}(\Phi + \Psi)] + f_R'(2\Phi - \Psi) = -2\tilde{\rho}(1 + c_s^2)v \quad (18)$$

$$\Phi - \Psi = -\frac{f_{RR}}{1 + f_R}\delta R \quad (19)$$

where  $\delta R$  is given by:

$$\delta R = -\frac{2}{a^2}[3\Psi'' + 6(\mathcal{H}' + \mathcal{H}^2)\Phi + 3\mathcal{H}(\Phi' + 3\Psi') - k^2(\Phi - 2\Psi)] \quad (20)$$

Finally, from the energy-momentum tensor conservation (15), we get to first order:

$$3\Psi'(1 + c_s^2) - \delta' + k^2(1 + c_s^2)v = 0 \quad (21)$$

and

$$\Phi + \frac{c_s^2}{1 + c_s^2}\delta + v' + \mathcal{H}v(1 - 3c_s^2) = 0 \quad (22)$$

for the temporal and spatial components respectively.

In a dust matter dominated universe, i.e.  $c_s^2 = 0$ , (21) and (22) can be combined to give

$$\delta'' + \mathcal{H}\delta' + k^2\Phi - 3\Psi'' - 3\mathcal{H}\Psi' = 0 \quad (23)$$

which will be very useful in future calculations.

## EVOLUTION OF DENSITY PERTURBATIONS

Our purpose is to derive a fourth order differential equation for matter density perturbation  $\delta$  alone. This can be performed by means of the following process:

Let us consider equations (16) and (18) for a matter dominated universe i.e.  $c_s^2 = 0$ , and combine them to express the potentials  $\Phi$  and  $\Psi$  in terms of  $\{\Phi', \Psi', \delta, \delta'\}$  by means of algebraic manipulations. The resulting expressions are the following

$$\begin{aligned} \Phi &= \frac{1}{\mathcal{D}(\mathcal{H}, k)} \left\{ [3(1 + f_R)\mathcal{H}(\Psi' + \Phi') \right. \\ &\quad + f_R'\Psi' + 2\tilde{\rho}\delta](1 + f_R)(\mathcal{H} - f_R') + [(1 + f_R)(\Phi' + \Psi') \\ &\quad + \frac{2\tilde{\rho}}{k^2}(\delta' - 3\Psi')][(1 + f_R)(-k^2 - 3\mathcal{H}') + 3f_R'\mathcal{H}] \left. \right\} \quad (24) \end{aligned}$$

and

$$\begin{aligned} \Psi &= \frac{1}{\mathcal{D}(\mathcal{H}, k)} \left\{ [-3(1 + f_R)\mathcal{H}(\Psi' + \Phi') - 3f_R'\Psi' \right. \\ &\quad - 2\tilde{\rho}\delta][(1 + f_R)\mathcal{H} + 2f_R'] - [(1 + f_R)(\Phi' + \Psi') \\ &\quad + \frac{2\tilde{\rho}}{k^2}(\delta' - 3\Psi')][(1 + f_R)(-k^2 + 3\mathcal{H}' - 6\mathcal{H}^2) - 9\mathcal{H}f_R'] \left. \right\} \quad (25) \end{aligned}$$

where

$$\mathcal{D}(\mathcal{H}, k) \equiv -6(1 + f_R)^2\mathcal{H}^3 + 3\mathcal{H}[f_R'^2 + 2(1 + f_R)^2\mathcal{H}'] + 3(1 + f_R)f_R'(-2\mathcal{H}^2 + k^2 + \mathcal{H}') \quad (26)$$

The second step will be to derive equations (24) and (25) with respect to  $\eta$  and obtain  $\Phi'$  and  $\Psi'$  algebraically in terms of  $\{\Phi'', \Psi'', \delta, \delta', \delta''\}$ . These last results can be substituted in equations (16) and (18) to obtain potentials  $\Phi$  and  $\Psi$  just in terms of  $\{\Phi'', \Psi'', \delta, \delta', \delta''\}$ . So at this stage we are able to express, but we do not do here explicitly, the following

$$\begin{aligned} \Phi &= \Phi(\Phi'', \Psi'', \delta, \delta', \delta'') \\ \Psi &= \Psi(\Phi'', \Psi'', \delta, \delta', \delta'') \\ \Phi' &= \Phi'(\Phi'', \Psi'', \delta, \delta', \delta'') \\ \Psi' &= \Psi'(\Phi'', \Psi'', \delta, \delta', \delta'') \end{aligned} \quad (27)$$

where we mean that the functions on the l.h.s. are algebraically dependent on the functions inside the parenthesis on the r.h.s.

The natural reasoning at this point would be to try to obtain the potentials second derivatives  $\{\Phi'', \Psi''\}$  in terms of  $\{\delta, \delta', \delta''\}$  by an algebraic process. The chosen equations to do so will be (23) and (19) first derivative with respect to  $\eta$ . In (23) it is necessary to substitute  $\Phi$  and  $\Psi'$  by the expressions obtained in (27) whereas (19) first derivative may be sketched as follows

$$\Phi' - \Psi' = -\frac{f_{RR}}{1 + f_R}\delta R' + \left[ \frac{f_{RR}f_R' - f_{RR}'(1 + f_R)}{(1 + f_R)^2} \right] \delta R \quad (28)$$

Before deriving, we are going to substitute  $\Psi''$  that appears on (19) by lower derivatives potentials  $\{\Phi, \Psi, \Phi', \Psi'\}$ ,  $\delta$  and its derivatives. To do so we consider (16) and (18) first derivatives with respect to  $\eta$  where the

quantity  $v$  has been previously substituted by its expression in (21). Following this process we may express  $\Psi''$  as follows

$$\Psi'' = \Psi''(\Phi, \Psi, \Phi', \Psi'; \delta, \delta', \delta'') \quad (29)$$

and now substituting in (19) we can derive that equation with respect to  $\eta$ . Solving a two algebraic equations system with equations (23) and (28) and introducing (27) we are able to express  $\{\Phi'', \Psi''\}$  in terms of  $\{\delta, \delta', \delta'', \delta'''\}$ .

$$\Phi'' = \Phi''(\delta, \delta', \delta'', \delta''') ; \Psi'' = \Psi''(\delta, \delta', \delta'', \delta''') \quad (30)$$

We substitute the results obtained in (30) straightforwardly in (27) in order to express  $\{\Phi, \Psi, \Phi', \Psi'\}$  in terms of  $\{\delta, \delta', \delta'', \delta'''\}$ . With the two potentials and its first derivatives as algebraic functions of  $\{\delta, \delta', \delta'', \delta'''\}$ , we performe the last step: We consider  $\Phi(\delta, \delta', \delta'', \delta''')$  and derive it with respect to  $\eta$ . The result should be equal to  $\Phi'(\delta, \delta', \delta'', \delta''')$  so we only need to express together these two results obtaining a fourth order differential equation for  $\delta$ . Note that this procedure is completely general to first order for scalar perturbations in the metric formalism for  $f(R)$  gravities.

Once this fourth order differential equation has been solved we may go backwards and by using the results for  $\delta$  we obtain  $\{\Phi'', \Psi''\}$  from (30) as functions of time. Analogously from (27) the behavior of the potentials  $\{\Phi, \Psi\}$  and their first derivatives could be determined.

The resulting equation for  $\delta$  can be written as follows:

$$\beta_{4,f} \delta^{iv} + \beta_{3,f} \delta''' + (\alpha_{2,\text{EH}} + \beta_{2,f}) \delta'' + (\alpha_{1,\text{EH}} + \beta_{1,f}) \delta' + (\alpha_{0,\text{EH}} + \beta_{0,f}) \delta = 0 \quad (31)$$

where the coefficients  $\beta_{i,f}$  ( $i = 1, \dots, 4$ ) involve terms with  $f'_R$  and  $f''_R$ , i.e. terms disappearing if we take  $f_R$  constant. Equivalently,  $\alpha_{i,\text{EH}}$  ( $i = 0, 1, 2$ ) contain terms coming from the linear part of  $f_0$  in  $R_0$ .

It is very useful to define the parameter  $\epsilon \equiv \mathcal{H}/k$  since it will allow us to perform a perturbative expansion of the previous coefficients  $\alpha$ 's and  $\beta$ 's in the sub-Hubble limit. Other dimensionless parameters which will be used are the following:  $\kappa_i \equiv \mathcal{H}'^{(i)}/\mathcal{H}^{i+1}$  ( $i = 1, 2, 3$ ) and  $f_i \equiv f_R^{(j)}/(\mathcal{H}^j f_R)$  ( $j = 1, 2$ ).

Expressing the  $\alpha$ 's and  $\beta$ 's coefficients with those dimensionless quantities we may write

$$\begin{aligned} \alpha_{i,\text{EH}} &= \sum_{j=1}^3 \alpha_{i,\text{EH}}^{(j)} \quad i = 0, 1, 2 \\ \beta_{i,f} &= \sum_{j=1}^7 \beta_{i,f}^{(j)} \quad i = 3, 4 \\ \beta_{i,f} &= \sum_{j=1}^8 \beta_{i,f}^{(j)} \quad i = 0, 1, 2 \end{aligned} \quad (32)$$

where two consecutives terms in each serie differ in  $\epsilon^2$  factor. The expressions for the coefficients are too long to be written explicitly. Instead, in the following sections we will show different approximated formulae useful in certain limits.

## VIABLE $f(R)$ THEORIES

Results obtained so far are valid for any  $f(R)$  theory. However, as mentioned in the introduction, this kind of models are severely constrained in order to provide consistent theories of gravity. In this section we review the main conditions [9]:

1.  $f_{RR} > 0$  for high curvatures [17]. This is the requirement for a classically stable high-curvature regime and the existence of a matter dominated phase in the cosmological evolution.
2.  $1 + f_R > 0$  for all  $R_0$ . This condition ensures the effective Newton's constant to be positive at all times and the graviton energy to be positive.
3.  $f_R < 0$  ensures ordinary General Relativity behaviour is recovered at early times. Together with the condition  $f_{RR} > 0$ , it implies that  $f_R$  should be negative and monotonically growing function of  $R_0$  in the range  $-1 < f_R < 0$ .
4.  $|f_R| \ll 1$  at recent epochs. This is imposed by local gravity tests [17], although it is still not clear what is the actual limit on this parameter. This condition also implies that the cosmological evolution at late times resembles that of  $\Lambda$ CDM. In any case, this constraint is not required if we are only interested in building models for cosmic acceleration.

## EVOLUTION OF SUB-HUBBLE MODES AND THE QUASI-STATIC APPROXIMATION

We are interested in the possible effects on the growth of density perturbations once they enter the Hubble radius in the matter dominated era. In the sub-Hubble limit  $\epsilon \ll 1$ , it can be seen that the  $\beta_{4,f}$  and  $\beta_{3,f}$  coefficients are suppressed by  $\epsilon^2$  with respect to  $\beta_{2,f}$ ,  $\beta_{1,f}$  and  $\beta_{0,f}$ , i.e., in this limit the equation for perturbations reduces to the following second order expression:

$$\delta'' + \mathcal{H}\delta' + \frac{(1+f_R)^5 \mathcal{H}^2 (-1+\kappa_1)(2\kappa_1-\kappa_2) - \frac{16}{a^8} f_{RR}^4 (\kappa_2-2) k^8 8\pi G \rho_0 a^2}{(1+f_R)^5 (-1+\kappa_1) + \frac{24}{a^8} f_{RR}^4 (1+f_R)(\kappa_2-2) k^8} \delta = 0 \quad (33)$$

where we have taken only the leading terms in the  $\epsilon$  expansion for the  $\alpha$  and  $\beta$  coefficients.

This expression can be compared with that usually considered in literature, obtained after performing strong simplifications in the perturbed equations - (16), (17), (18), (19), (21) and (22) - by neglecting time derivatives of  $\Phi$  and  $\Psi$  potentials, (see [14]). Thus in [10] and [18] they obtain:

$$\delta'' + \mathcal{H}\delta' - \frac{1 + 4\frac{k^2}{a^2} \frac{f_{RR}}{1+f_R}}{1 + 3\frac{k^2}{a^2} \frac{f_{RR}}{1+f_R}} \frac{\tilde{\rho}\delta}{1+f_R} = 0 \quad (34)$$

This approximation has been considered as too aggressive in [11] since neglecting time derivatives can remove important information about the evolution.

Note also that there exists a difference in a power  $k^8$  between those terms coming from the  $f$  part and those coming from the  $EH$  part in (33). This result differs from that in the quasi-static approximation where difference is in a power  $k^2$  according to (34).

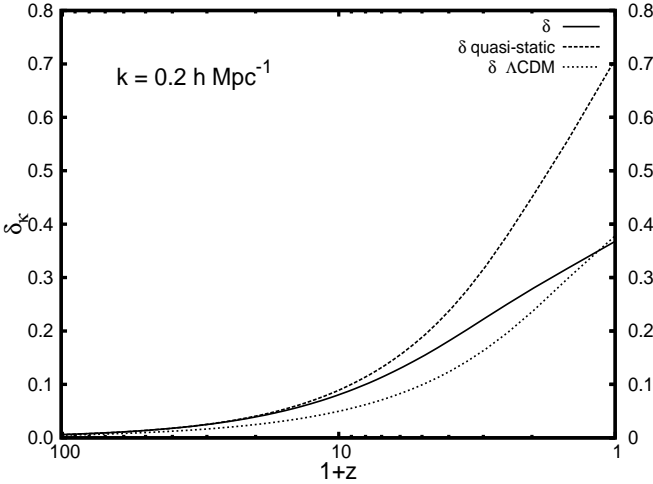


Figure 1:  $\delta_k$  with  $k = 0.2h\text{Mpc}^{-1}$  for  $f_{test}(R)$  model and  $\Lambda\text{CDM}$ . Both, standard quasi-static evolution and equation (33) have been plotted in the redshift range from 100 to 0.

In order to compare the evolution for both equations, we have considered a specific function  $f_{test}(R) = -4R^{0.63}$ , where  $H_0^2$  units have been used, which gives rise to a matter era followed by a late time accelerated phase with the correct deceleration parameter today. Initial conditions in the matter era were given at redshift  $z = 485$  where the  $EH$  part was dominant. Results, for

$k = 600H_0$  are presented in figure (1). We see that, as expected, both expressions give rise to the same evolutions at early times (large redshifts) where they also agree with the standard  $\Lambda\text{CDM}$  evolution. However, at late times the quasi-static approximation fails to correctly describe the evolution of perturbations.

Notice that the model example satisfies all the viability conditions described in the previous section except for the local gravity tests. As we will show in the next section, it is precisely this last condition  $|f_R| \ll 1$  what will ensure the validity of the quasi-static approximation.

### Recovering the quasi-static limit

We will now restrict ourselves to models satisfying all the viability conditions, including  $|f_R| \ll 1$ .

In Appendix I we have reproduced all the  $\alpha$ 's and the first four  $\beta$ 's coefficients for each  $\delta$  term in (31). These are the dominant ones for sub-Hubble modes (i.e.  $\epsilon \ll 1$ ) once the condition  $|f_R| \ll 1$  has been imposed. Thus, keeping only  $\sum_{j=1}^4 \beta_{i=0,\dots,4,f}^{(j)}$  and  $\alpha_{i=0,1,2,EH}^{(1)}$  as the relevant contributions for the general coefficients, the full differential equation (31) can be simplified as

$$c_4 \delta^{iv} + c_3 \delta''' + c_2 \delta'' + c_1 \delta' + c_0 \delta = 0 \quad (35)$$

where coefficients  $c$ 's are written in Appendix II.

We see that indeed in the sub-Hubble limit the  $c_4$  and  $c_3$  coefficients are negligible and the equation can be reduced to a second order expression.

As a consistency check, we find that both, in a matter dominated universe and in  $\Lambda\text{CDM}$  all  $\beta$  coefficients vanish identically since  $f_1, f_2 \equiv 0$ . For these cases, equation (31) becomes equation (6) as expected. For instance, in the pure matter dominated case, coefficients  $\kappa$ 's are constant and they take the following values  $\kappa_1 = -1/2$ ,  $\kappa_2 = 1/2$ ,  $\kappa_3 = -3/4$  and  $\kappa_4 = 3/2$ .

Another important feature from our results is that, in general, without imposing  $|f_R| \ll 1$ , the quotient  $(\alpha_{1,EH} + \beta_{1,f}) / (\alpha_{2,EH} + \beta_{2,f})$  is not always equal to  $\mathcal{H}$ . In fact only the quotients  $\alpha_{1,EH}^{(1)} / \alpha_{2,EH}^{(1)}$  and  $\beta_{1,f}^{(1)} / \beta_{2,f}^{(1)}$  are identically equal to  $\mathcal{H}$  what is in agreement with the  $\delta'$  coefficient in (6). However for our approximated expressions it is true that  $c_1/c_2 \equiv \mathcal{H}$ .

From expressions in Appendix II, the second order equation for  $\delta$  becomes

$$\delta'' + \mathcal{H}\delta' - \frac{4}{3} \frac{\left[\frac{6f_{RR}k^2}{a^2} + \frac{9}{4}(1 - \sqrt{1 - \frac{8}{9} \frac{2\kappa_1 - \kappa_2}{-2 + \kappa_2}})\right] \left[\frac{6f_{RR}k^2}{a^2} + \frac{9}{4}(1 + \sqrt{1 - \frac{8}{9} \frac{2\kappa_1 - \kappa_2}{-2 + \kappa_2}})\right]}{\left[\frac{6f_{RR}k^2}{a^2} + \frac{5}{2}(1 - \sqrt{1 - \frac{24}{25} \frac{-1 + \kappa_1}{-2 + \kappa_2}})\right] \left[\frac{6f_{RR}k^2}{a^2} + \frac{5}{2}(1 + \sqrt{1 - \frac{24}{25} \frac{-1 + \kappa_1}{-2 + \kappa_2}})\right]} (1 - \kappa_1) \mathcal{H}^2 \delta = 0 \quad (36)$$

which can also be written as:

$$\delta'' + \mathcal{H}\delta' - \frac{4}{3} \frac{\left(\frac{6f_{RR}k^2}{a^2} + \frac{9}{4}\right)^2 - \frac{81}{16} + \frac{9}{2} \frac{2\kappa_1 - \kappa_2}{-2 + \kappa_2}}{\left(\frac{6f_{RR}k^2}{a^2} + \frac{5}{2}\right)^2 - \frac{25}{4} + 6 \frac{-1 + \kappa_1}{-2 + \kappa_2}} (1 - \kappa_1) \mathcal{H}^2 \delta = 0 \quad (37)$$

Note that the quasi-static expression (34) is only recovered in the matter era (i.e. for  $\mathcal{H} = 2/\eta$ ) or for a pure  $\Lambda$ CDM evolution for the background dynamics. Nevertheless in the considered limit  $|f_R| \ll 1$  it can be proven using the background equations of motion that

$$1 + \kappa_1 - \kappa_2 \approx 0 \quad (38)$$

and therefore  $2\kappa_1 - \kappa_2 \approx -2 + \kappa_2 \approx -1 + \kappa_1$  what allows to simplify expression (37) to approximately become (34). This is nothing but the fact that for viable models the background evolution resembles that of  $\Lambda$ CDM [9].

In other words, although for general  $f(R)$  functions the quasi-static approximation is not justified, for those viable functions describing the present phase of accelerated expansion and satisfying local gravity tests, it gives a correct description for the evolution of perturbations.

### SOME PROPOSED MODELS

In order to check the results obtained in the previous section, we propose two particular  $f(R)$  theories which allow us to determine - at least numerically - all the quantities involved in the calculations and therefore to obtain solutions for (31). As commented before, for viable models the background evolution resemble that of  $\Lambda$ CDM at low redshifts and that of a matter dominated universe at high redshifts, i.e. the quantity  $(R + f(R))/R$  tends to one in the high curvature regime. Nevertheless the  $f(R)$  contribution gives the dominant contribution to the gravitational action for small curvatures and therefore it may explain the cosmological acceleration. For the sake of concreteness we will fix the model parameters imposing a deceleration parameter today  $q_0 \approx -0.6$ .

Thus, our first model (A) will be:  $f(R) = c_1 R^p$ . According to the results presented in [12] and [19] viable models of this type include both matter dominated and late-time accelerated universe provided the parameters satisfy  $c_1 < 0$  and  $0 < p < 1$ . We have chosen  $c_1 = -4.3$  and  $p = 0.01$  in  $H_0^2$  units. This choice does verify all the viability conditions, including  $|f_R| \ll 1$  today. For the second model (B):  $f(R) = \frac{1}{c_1 R^{e_1 + c_2}}$ , we have chosen

$c_1 = 2.5 \cdot 10^{-4}$ ,  $e_1 = 0.3$  and  $c_2 = -0.22$  also in the same units.

For each model, we compare our result (36) with the standard  $\Lambda$ CDM and the quasi-static approximation (34) (see Figs. 2 and 3). In both cases, the initial conditions are given at redshift  $z = 1000$  where  $\delta$  is assumed to behave as in a matter dominated universe, i.e.  $\delta_k(\eta) \propto a(\eta)$  with no  $k$ -dependence. We see that for both models, the quasi-static approximation gives a correct description for the evolution which clearly deviates from the  $\Lambda$ CDM case.

In figure (4) the density contrast evaluated today was plotted as a function of  $k$  for both models. The growing dependence of  $\delta$  with respect to  $k$  is verified. This modified  $k$ -dependence with respect to the standard matter dominated universe could give rise to observable consequences in the matter power spectrum, as shown in [13], and could be used to constrain or even discard  $f(R)$  theories for cosmic acceleration.

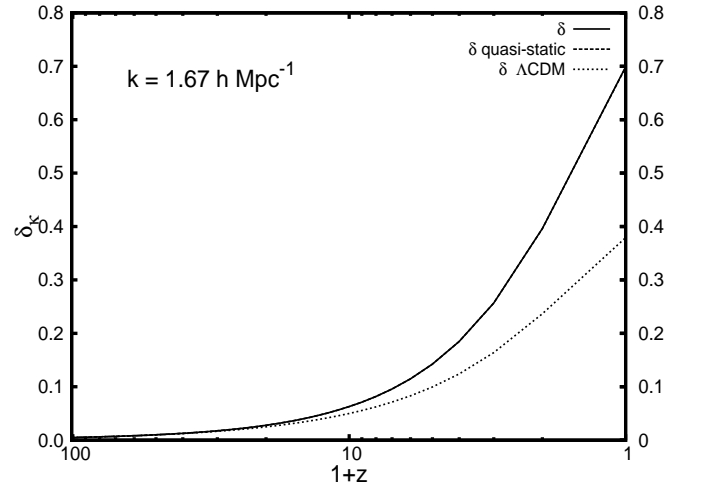


Figure 2:  $\delta_k$  with  $k = 1.67 h \text{Mpc}^{-1}$  for  $f(R)$  model **A** evolving according to (36),  $\Lambda$ CDM and quasi-static approximation given by equation (34) in the redshift range from 1000 to 0. The quasi-static evolution is indistinguishable from that coming from (36), but diverges from  $\Lambda$ CDM behaviour as  $z$  decreases.

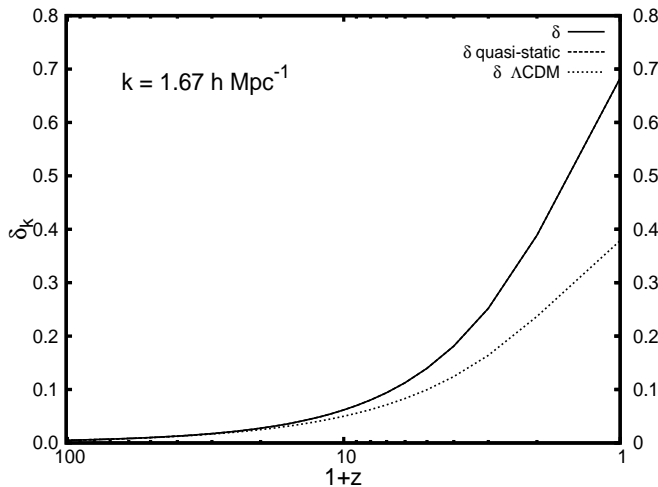


Figure 3:  $\delta_k$  with  $k = 1.67 h \text{ Mpc}^{-1}$  for  $f(R)$  model **B** evolving according to (36),  $\Lambda\text{CDM}$  and quasi-static evolution given by equation (34) in the redshift range from 1000 to 0. The quasi-static evolution is indistinguishable from that coming from (36), but diverges from  $\Lambda\text{CDM}$  behaviour as  $z$  decreases.

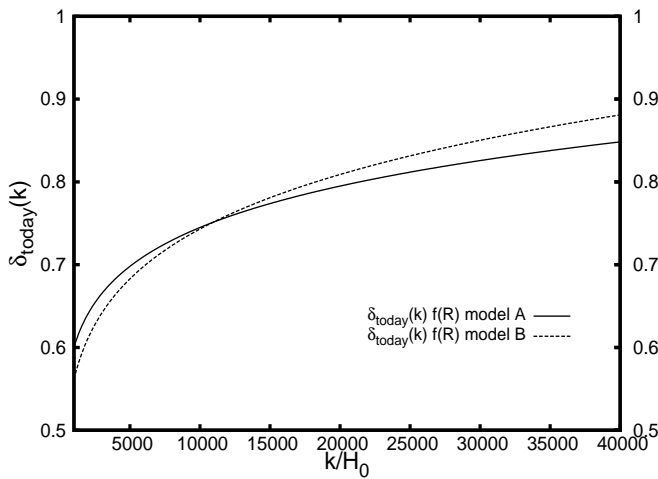


Figure 4: Scale dependence of  $\delta_k$  evaluated today ( $z = 0$ ) for  $k/H_0$  in the range from 1000 to 40000.

## CONCLUSIONS

In this work we have studied the evolution of matter density perturbations in  $f(R)$  theories of gravity. We have presented a completely general procedure to obtain the exact fourth-order differential equation for the evolution of perturbations. We have shown that for sub-Hubble modes, the expression reduces to a second order equation. We have compared this result with that obtained within quasi-static approximation used in the literature and found that for arbitrary  $f(R)$  functions, such an approximation is not justified.

However, if we limit ourselves to theories for which

$|f_R| \ll 1$  today, then the perturbative calculation for sub-Hubble modes requires to take into account, not only the first terms, but also higher-order terms in  $\epsilon = \mathcal{H}/k$ . In that case, the resummation of such terms modifies the equation which can be seen to be equivalent to the quasi-static case but only if the universe expands as in a matter dominated phase or in a  $\Lambda\text{CDM}$  model. Finally, the fact that for models with  $|f_R| \ll 1$  the background behaves today precisely as that of  $\Lambda\text{CDM}$  makes the quasi-static approximation correct in those cases.

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## APPENDICES

### Appendix I: $\alpha'$ s and $\beta'$ s coefficients

Coefficients for  $\delta^{iv}$  term:

$$\begin{aligned}
 \beta_{4,f}^{(1)} &\simeq 8f_R^4(1+f_R)^6 f_1^4 \epsilon^2 \\
 \beta_{4,f}^{(2)} &\simeq 72f_R^3 f_1^3 \epsilon^4 (-2 + \kappa_2) \\
 \beta_{4,f}^{(3)} &\simeq 216f_R^2 f_1^2 \epsilon^6 (-2 + \kappa_2)^2 \\
 \beta_{4,f}^{(4)} &\simeq 216f_R f_1 \epsilon^8 (-2 + \kappa_2)^3
 \end{aligned} \tag{39}$$

Coefficients for  $\delta'''$  term:

$$\begin{aligned}
 \beta_{3,f}^{(1)} &\simeq 8f_R^4(1+f_R)^5 f_1^4 \mathcal{H} \epsilon^2 [3 + f_R(3 + f_1)] \\
 \beta_{3,f}^{(2)} &\simeq 6f_R^3 f_1^2 \mathcal{H} \epsilon^4 \{8f_2(-2 + \kappa_2) + 4f_1[12\kappa_1 + 9\kappa_2 - 2(9 + \kappa_3)]\} \\
 \beta_{3,f}^{(3)} &\simeq -72f_R^2 f_1 \mathcal{H} \epsilon^6 (-2 + \kappa_2)[-4f_2(-2 + \kappa_2) + f_1(19 - 23\kappa_1 - 10\kappa_2 + 4\kappa_3)] \\
 \beta_{3,f}^{(4)} &\simeq -216f_R \mathcal{H} \epsilon^8 (-2 + \kappa_2)^2 [-2f_2(-2 + \kappa_2) + f_1(7 - 11\kappa_1 - 4\kappa_2 + 2\kappa_3)]
 \end{aligned} \tag{40}$$

Coefficients for  $\delta''$  term:

$$\begin{aligned}
 \alpha_{2,EH}^{(1)} &= 432(1+f_R)^{10} \mathcal{H}^2 \epsilon^8 (-1 + \kappa_1)(-2 + \kappa_2)^3 \\
 \alpha_{2,EH}^{(2)} &= 1296(1+f_R)^{10} \mathcal{H}^2 \epsilon^{10} (-1 + \kappa_1)^2 (-2 + \kappa_2)^3 \\
 \alpha_{2,EH}^{(3)} &= 3888(1+f_R)^{10} \mathcal{H}^2 \epsilon^{12} (-1 + \kappa_1)^2 (-2 + \kappa_2)^3 \\
 \beta_{2,f}^{(1)} &\simeq 8f_R^4(1+f_R)^6 f_1^4 \mathcal{H}^2 \\
 \beta_{2,f}^{(2)} &\simeq 88f_R^3 f_1^3 \mathcal{H}^2 \epsilon^2 (-2 + \kappa_2) \\
 \beta_{2,f}^{(3)} &\simeq 24f_R^2 f_1^2 \mathcal{H}^2 \epsilon^4 (-2 + \kappa_2)(-28 + 2\kappa_1 + 13\kappa_2) \\
 \beta_{2,f}^{(4)} &\simeq 72f_R f_1 \mathcal{H}^2 \epsilon^6 (-2 + \kappa_2)^2 (-14 + 4\kappa_1 + 5\kappa_2)
 \end{aligned} \tag{41}$$

Coefficients for  $\delta'$  term:

$$\begin{aligned}
 \alpha_{1,EH}^{(1)} &= 432(1+f_R)^{10} \mathcal{H}^3 \epsilon^8 (-1 + \kappa_1)(-2 + \kappa_2)^3 \\
 \alpha_{1,EH}^{(2)} &= 2592(1+f_R)^{10} \mathcal{H}^3 \epsilon^{10} (-1 + \kappa_1)^2 (-2 + \kappa_2)^3 \\
 \alpha_{1,EH}^{(3)} &= -7776(1+f_R)^{10} \mathcal{H}^3 \epsilon^{12} (-1 + \kappa_1)^3 (-2 + \kappa_2)^3 \\
 \beta_{1,f}^{(1)} &\simeq 8f_R^4(1+f_R)^6 f_1^4 \mathcal{H}^3 \\
 \beta_{1,f}^{(2)} &\simeq 88f_R^3 f_1^3 \mathcal{H}^3 \epsilon^2 (-2 + \kappa_2) \\
 \beta_{1,f}^{(3)} &\simeq 24f_R^2 f_1^2 \mathcal{H}^3 \epsilon^4 (-2 + \kappa_2)(-28 + 2\kappa_1 + 13\kappa_2) \\
 \beta_{1,f}^{(4)} &\simeq 72f_R f_1 \mathcal{H}^3 \epsilon^6 (-2 + \kappa_2)^2 (-14 + 4\kappa_1 + 5\kappa_2)
 \end{aligned} \tag{42}$$



Coefficients for  $\delta$  term:

$$\begin{aligned}
\alpha_{0,\text{EH}}^{(1)} &= 432(1+f_R)^{10}\mathcal{H}^4\epsilon^8(-1+\kappa_1)(2\kappa_1-\kappa_2)(-2+\kappa_2)^3 \\
\alpha_{0,\text{EH}}^{(2)} &= 1296(1+f_R)^{10}\mathcal{H}^4\epsilon^{10}(-1+\kappa_1)^2(-1+4\kappa_1-\kappa_2)(-2+\kappa_2)^3 \\
\alpha_{0,\text{EH}}^{(3)} &= 3888(1+f_R)^{10}\mathcal{H}^4\epsilon^{12}(-1+\kappa_1)^2(2\kappa_1^2-\kappa_2)(-2+\kappa_2)^3 \\
\beta_{0,f}^{(1)} &\simeq -\frac{16}{3}f_R^4(1+f_R)^5f_1^4\mathcal{H}^4[2+f_R(2+2f_1-f_2-2\kappa_1)-2\kappa_1] \\
\beta_{0,f}^{(2)} &\simeq 112f_R^3f_1^3\mathcal{H}^4\epsilon^2(-1+\kappa_1)(-2+\kappa_2) \\
\beta_{0,f}^{(3)} &\simeq 48f_R^2f_1^2\mathcal{H}^4\epsilon^4(-1+\kappa_1)(-2+\kappa_2)(-16+2\kappa_1+7\kappa_2) \\
\beta_{0,f}^{(4)} &\simeq 144f_Rf_1\mathcal{H}^4\epsilon^6(-1+\kappa_1)(-2+\kappa_2)^2(-6+4\kappa_1+\kappa_2)
\end{aligned} \tag{43}$$

## Appendix II: $c'$ s coefficients

$$\begin{aligned}
c_4 &= -f_Rf_1[-f_Rf_1k^2-3\mathcal{H}^2(-2+\kappa_2)]^3 \\
c_3 &= -3f_R\mathcal{H}[-f_Rf_1k^2-3\mathcal{H}^2(-2+\kappa_2)]\{f_R^2f_1^3k^4+6f_2\mathcal{H}^4(-2+\kappa_2)^2+f_1\mathcal{H}^2(-2+\kappa_2)[2f_Rf_2k^2+ \\
&\quad 3\mathcal{H}^2(-7+11\kappa_1+4\kappa_2-2\kappa_3)]+2f_Rf_1\mathcal{H}^2k^2(-6+6\kappa_1+3\kappa_2-\kappa_3)\} \\
c_2 &= [-f_Rf_1k^2-3\mathcal{H}^2(-2+\kappa_2)]^2[f_R^2f_1^2k^4+5f_Rf_1\mathcal{H}^2k^2(-2+\kappa_2)+6\mathcal{H}^4(-1+\kappa_1)(-2+\kappa_2)] \\
c_1 &= \mathcal{H}c_2 \\
c_0 &= \frac{2}{3}\mathcal{H}^2(-1+\kappa_1)[-f_Rf_1k^2-3\mathcal{H}^2(-2+\kappa_2)]^2[2f_R^2f_1^2k^4+9f_Rf_1\mathcal{H}^2k^2(-2+\kappa_2)+9\mathcal{H}^4(2\kappa_1-\kappa_2)(-2+\kappa_2)]
\end{aligned} \tag{44}$$


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